Five-loop additive renormalization in the ϕ^4 theory and amplitude functions of the minimally renormalized specific heat in three dimensions

S. A. Larin*, M. Mönnigmann, M. Strösser, and V. Dohm

Institut für Theoretische Physik, Technische Hochschule Aachen, D–52056 Aachen, Germany

Abstract

We present an analytic five-loop calculation for the additive renormalization constant $A(u, \epsilon)$ and the associated renormalization-group function $B(u)$ of the specific heat of the O(n) symmetric ϕ^4 theory within the minimal subtraction scheme. We show that this calculation does not require new five-loop integrations but can be performed on the basis of the previous five-loop calculation of the four-point vertex function combined with an appropriate identification of symmetry factors of vacuum diagrams. We also determine the amplitude functions of the specific heat in three dimensions for $n = 1, 2, 3$ above T_c and for $n = 1$ below T_c up to five-loop order. Accurate results are obtained from Borel resummations of $B(u)$ for $n = 1, 2, 3$ and of the amplitude functions for $n = 1$. Previous conjectures regarding the smallness of the resummed higher-order contributions are confirmed. Borel resummed universal amplitude ratios A^+/A^- and a_c^+/a_c^- are calculated for $n = 1$.

05.70.Jk, 11.10.Gh, 64.60.Ak, 67.40.Kh

*Permanent address: Institute for Nuclear Research of the Russian Academy of Science, 60^{th} October Anniversary Prospect 7-A, Moscow 117312, Russia

I. INTRODUCTION

One of the fundamental achievements of the renormalization-group (RG) theory of critical phenomena is the identification of universality classes in terms of the dimensionality d of the system and the number n of components of the order parameter [1]. Specifically, RG theory predicts that the critical exponents, certain amplitude ratios and scaling functions are universal quantities that do not depend, e.g., on the strength of the interaction or on thermodynamic variables (such as the pressure). The superfluid transition of ^{4}He belongs to the $d = 3$, $n = 2$ universality class and provides a unique opportunity for an experimental test of the universality prediction by means of measurements of the critical behavior at various pressures P along the λ-line $T_\lambda(P)$. Early tests have been performed by Ahlers and collaborators and consistency with the universality prediction was found within the experimental resolution [2]. At a significantly higher level of accuracy, the superfluid density and the specific heat (or, equivalently, thermal expansion coefficient) above and below $T_\lambda(P)$ are planned to be measured in the Superfluid Universality Experiment (SUE) [3] under microgravity conditions or at reduced gravity in the low-gravity simulator [4]. As demonstrated recently [5], this would allow to perform measurements up to $|t| \simeq 10^{-9}$ in the reduced temperature $t = (T - T_\lambda(P))/T_\lambda(P)$.

On the theoretical side, the corresponding challenge is to calculate as accurately as possible the properties of the O(n) symmetric ϕ^4 model in three dimensions. To extract the leading critical exponents from the experimental data and to demonstrate their universality at a highly quantitative level requires detailed knowledge on the ingredients of a nonlinear RG analysis [6]. They include not only the well-known RG exponent functions of the ϕ^4 model whose fixed point values determine the critical exponents but also the less well-known amplitude functions [7–12] which contain the information about universal ratios of leading and subleading amplitudes [1].

The existing theoretical predictions on the critical exponents [13] within the minimal subtraction scheme [14,15] are based on field-theoretic calculations to five-loop order [16–19]

and Borel resummation. By contrast, the present theoretical knowledge of the amplitude ratios for $n > 1$ below T_c is based only on low-order (mainly 1- and 2-loop) calculations which imply an uncertainty at the level of at least 10–30% [1]. It has therefore been proposed [20] to significantly reduce this uncertainty by performing new higher-order field-theoretic calculations and Borel resummations of various amplitude functions in three dimensions.

Both conceptual and computational steps towards this goal have already been performed. The conceptual progress includes the demonstration that the $d = 3$ field theory suggested by Parisi [21] can well be realized within the minimal subtraction scheme at $d = 3$ [7–9] by incorporating Symanzik's non-vanishing mass shift [22] and that spurious Goldstone singularities for $n > 1$ below T_c can well be treated within this approach [12] by using an appropriately defined pseudo-correlation length [9]. The computational steps include the determination of the amplitude functions $F_+(u)$ and $F_-(u)$ of the specific heat in three dimensions above T_c for $n = 1, 2, 3$ [10] and below T_c for $n = 1$ [11], respectively, up to five-loop order, and their Borel resummation. These calculations, however, were not yet complete because of an approximation regarding the additive renormalization $A(u, \varepsilon)$ of the specific heat and the associated RG function $B(u)$. Due to the lack of knowledge in the literature about higher-order terms, $A(u, \varepsilon)$ and $B(u)$ were approximated by their two-loop expressions. Although the good agreement between low-order $d = 3$ perturbation results [7,23,24] and accurate experiments [2,25,26] provided some indication for the smallness of the effect of the higher-order terms of $B(u)$, no reliable estimate could be given for the remaining uncertainty of $F_\pm(u)$ which could well be of relevance at the level of accuracy anticipated in future experiments [3]. Furthermore we recall that any inaccuracy of $B(u)$ enters not only the formulas [9] for several universal amplitude ratios but also the formulas needed to determine the effective coupling $u(l)$ from the specific heat [7,23,24].

It is the purpose of the present paper to provide the missing information on the higher-order terms of $A(u, \varepsilon)$ and $B(u)$ by means of a new five-loop calculation. We shall show that the analytic calculation of $A(u, \varepsilon)$ and $B(u)$ can be directly related to the previous calculations [16–19] of the four-point vertex function. This provides the crucial simplification

that no new evaluations of three-, four- and five-loop integrals are necessary but that only a new determination of symmetry and O(n) group factors of vacuum diagrams is sufficient.

Using the five-loop expression of $B(u)$ we are in the position to determine the correct higher-order terms of the minimally renormalized amplitude functions $F_+(u)$ for $n = 1, 2, 3$ and $F_-(u)$ for $n = 1$ on the basis of previous work [27–30] where a different renormalization scheme was used. The new coefficients of the higher-order terms of $F_+(u)$ turn out to differ considerably from the previous approximate coefficients [10] whereas the coefficients of $F_-(u)$ are only weakly affected by the new higher-order terms of $B(u)$.

We also perform new Borel resummations of $B(u)$ for $n = 1, 2, 3$ as well as of $F_-(u)$ and of $F_-(u) - F_+(u)$ for $n = 1$. It turns out that the result of the Borel resummation for $B(u)$ including the new terms up to five-loop order differs from the two-loop result $B(u) = n/2 + O(u^2)$ by only less than 1% at the fixed point. For the amplitude functions, our new Borel resummation results differ from the previous ones [10,11] by about 1% for F_- and by less than 0.1% for $F_- - F_+$ at the fixed point. This is a nontrivial and important confirmation of the previous conjectures about the smallness of resummed higher-order contributions [9–11]. As a first application, we calculate the universal ratios A^+/A^- and a_c^+/a_c^- of the leading and subleading amplitudes of the specific heat above and below T_c for $n = 1$ and $d = 3$. Fully quantitative calculations for $n > 1$ have to be postponed until higher-order results for $F_-(u)$ are available.

II. ADDITIVE RENORMALIZATION OF THE SPECIFIC HEAT

The O(n) symmetric ϕ^4 model is defined by the usual Landau-Ginzburg-Wilson functional

$$\mathcal{H}\{\vec{\phi}_0(\mathbf{x})\} = \int_V \mathrm{d}^d x \left(\frac{1}{2} r_0 \phi_0^2 + \frac{1}{2} \sum_i (\nabla \phi_{0i})^2 + u_0 (\phi_0^2)^2 - \vec{h}_0 \cdot \vec{\phi}_0 \right) \tag{2.1}$$

for the n-component field $\vec{\phi}_0(\mathbf{x}) = (\phi_{01}(\mathbf{x}), \dots, \phi_{0n}(\mathbf{x}))$ where

$$r_0 = r_{0\mathrm{c}} + a_0 t \,, \quad t = (T - T_\mathrm{c})/T_\mathrm{c} \,, \tag{2.2}$$

and $\vec{h}_0 = (h_0, 0, \ldots, 0)$. The Gibbs free energy per unit volume (divided by $k_B T$) is

$$F_0(r_0, u_0, h_0) = -V^{-1} \ln \int \mathcal{D}\vec{\phi}_0 \, \exp(-\mathcal{H}) \,. \tag{2.3}$$

We shall consider the bulk limit $V \to \infty$. We are interested in the specific heat $\mathring{C}^{\pm}$ per unit volume at vanishing external field $h_0 = 0$ (divided by Boltzmann's constant k_B) where $\pm$ refers to $T > T_\mathrm{c}$ and $T < T_\mathrm{c}$, respectively. Near T_c, $\mathring{C}^{\pm}$ is determined by [9]

$$\mathring{C}^{\pm} = C_B - T_\mathrm{c}^2 \frac{\partial^2}{\partial T^2} F_0(r_0, u_0, 0) = C_B - a_0^2 \frac{\partial^2}{\partial r_0^2} F_0(r_0, u_0, 0) \tag{2.4}$$

where C_B is an analytic background term. Alternatively the Helmholtz free energy per unit volume $\Gamma_0(r_0, u_0, M_0) = F(r_0, u_0, h_0) + h_0 M_0$ with $M_0 \equiv \langle \phi_{01} \rangle$ determines $\mathring{C}^{\pm}$ in the $h_0 \to 0$ limit according to

$$\mathring{C}^{\pm} = C_B - a_0^2 \frac{\mathrm{d}^2}{\mathrm{d}r_0^2} \Gamma_0 \left(r_0, u_0, M_0(r_0, u_0) \right) \,. \tag{2.5}$$

The perturbative expression for $\Gamma_0(r_0, u_0, M_0)$ is obtained from the negative sum of all one-particle irreducible (1 PI) vacuum diagrams [15]. The perturbative expression for $\mathring{C}^{\pm}$ is then determined by the vertex functions $\mathring{\Gamma}_{\pm}^{(2,0)} = \mathrm{d}^2 \Gamma_0 / \mathrm{d}r_0^2$ which we consider as functions of appropriately defined correlation lengths ξ_+ and ξ_- above and below T_c [9,12],

$$\mathring{C}^{\pm} = C_B - a_0^2 \mathring{\Gamma}_{\pm}^{(2,0)}(\xi_{\pm}, u_0, d) \,. \tag{2.6}$$

A description of the critical behavior requires to turn to the renormalized vertex functions

$$\Gamma_{\pm}^{(2,0)}(\xi_{\pm}, u, \mu, d) = Z_r^2 \mathring{\Gamma}_{\pm}^{(2,0)}(\xi_{\pm}, \mu^{\epsilon} Z_u Z_\phi^{-2} A_d^{-1} u, d) - \frac{1}{4} \mu^{-\epsilon} A_d A(u, \epsilon) \,. \tag{2.7}$$

We work at infinite cutoff using the prescriptions of dimensional regularization and minimal subtraction at fixed dimension $2 < d < 4$ without employing the $\epsilon = 4 - d$ expansion [7–9]. The Z-factors are introduced as

$$r = Z_r^{-1}(r_0 - r_{0\mathrm{c}}), \quad u = \mu^{-\epsilon} A_d Z_u^{-1} Z_\phi^2 u_0, \quad \vec{\phi} = Z_\phi^{-1/2} \vec{\phi}_0 \tag{2.8}$$

where the geometric factor

$$A_d = \Gamma(3 - d/2)\, 2^{2-d} \pi^{-d/2} (d-2)^{-1} \tag{2.9}$$

becomes $A_3 = (4\pi)^{-1}$ for $d = 3$ and $A_4 = (8\pi^2)^{-1}$ for $d = 4$. These Z-factors $Z_i(u, \epsilon)$ and the associated field-theoretic functions [8]

$$\zeta_r(u) = \mu\, \partial_\mu \ln Z_r(u, \epsilon)^{-1}\Big|_0 \,, \tag{2.10}$$

$$\zeta_\phi(u) = \mu\, \partial_\mu \ln Z_\phi(u, \epsilon)^{-1}\Big|_0 \,, \tag{2.11}$$

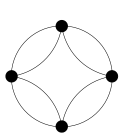
$$\beta_u(u, \epsilon) = -\epsilon u + \tilde{\beta}(u) = u \left[-\epsilon + \mu\, \partial_\mu (Z_u^{-1} Z_\phi^2)\Big|_0 \right] \,, \tag{2.12}$$

are known up to five-loop order [17–19].

The main quantity of interest in the present paper is the renormalization constant $A(u, \epsilon)$ in Eq. (2.7) which absorbs the additive poles of both $\mathring{\Gamma}_+^{(2,0)}$ and $\mathring{\Gamma}_-^{(2,0)}$. Previously [10,11] $A(u, \epsilon)$ was employed only in its two-loop form [7]

$$A(u, \epsilon) = -2n \frac{1}{\epsilon} - 8n(n+2) \frac{u}{\epsilon^2} + O(u^2) \,. \tag{2.13}$$

Here we report on a calculation of $A(u, \epsilon)$ up to five-loop order. We would like to stress that this calculation does not require new five-loop integrations but can be performed on the basis of the previous five-loop calculation [17–19] of the four-point vertex function combined with an appropriate identification of symmetry and O(n) group factors of vacuum diagrams which are shown in Fig. 1. Their negative sum determines the Helmholtz free energy Γ_0 up to five-loop order. For the present purpose of determining the pole terms at $d = 4$ it suffices to consider only the case $r_0 > 0$ and $M_0 = 0$ where only four-point vertices exist. The diagrams are labeled (1) in one-loop order, (2) in two-loop order, (3), (4) in three-loop order, (5)–(8) in four-loop order and (9)–(18) in five-loop order. The analytic expression of an m-loop diagram (i) is given by the product of the coupling $(-u_0)^{m-1}$, the symmetry factor $S^{(i)}$, the O(n) group factor $G^{(i)}(n)$ and the momentum integral expression $I^{(i)}(r_0, \epsilon)$. Thus the structure of the diagrammatic expression of a typical diagram, e.g., (16) is



$$= \quad S^{(16)} (-u_0)^4\, G^{(16)}\, I^{(16)}(r_0, \epsilon) \tag{2.14}$$

$$= \quad 2592\,(-u_0)^4\,\frac{n^4+8n^3+32n^2+40n}{81}$$
$$\times \int_{p_1}\int_{p_2}\int_{p_3}\int_{p_4}\int_{p_5} G_1\cdot G_2\cdot G_{1+2-3}\cdot G_3\cdot G_{1+2-4}\cdot G_4\cdot G_{1+2-5}\cdot G_5 \qquad (2.15)$$

with $\int_p \equiv (2\pi)^{-d}\int \mathrm{d}^d p$ and the propagators $G_{i\pm j}\equiv (r_0+|\mathbf{p}_i\pm\mathbf{p}_j|^2)^{-1}$. The symmetry and group factors are listed in Table I.

To calculate the additive renormalization constant $A(u,\epsilon)$ one needs to calculate those ultraviolet $d=4$ pole terms of the diagrams contributing to $\Gamma_{0+}^{(2,0)}$ that are left after subtraction of subdivergences. One obtains $\Gamma_{0+}^{(2,0)}$ by taking two derivatives of Γ_0 with respect to r_0. The analytic calculation of the poles of the diagrams for $\Gamma_{0+}^{(2,0)}$ is identical to that carried out previously [17–19] for the four-point vertex function $\Gamma_0^{(0,4)}$. To see this, one should take into account that in the minimal subtraction scheme [14] the ultraviolet pole terms specified above do not depend on r_0. Then by using the method of infrared rearrangement [34] one can nullify r_0 and introduce for each diagram a new fictitious external momentum to regularize infrared divergences. Then one can see that only a particular subset of those diagrams of $\Gamma_0^{(0,4)}$ are relevant in the present context, namely those where the four external legs are connected to each diagram through only two four-point vertices (rather than three four-point vertices or four four-point vertices). The details of the calculation are presented in Appendix A.

The result reads

$$A(u,\epsilon)=\sum_{m=1}^{5}A^{(m)}(u,\epsilon)+O(u^5) \qquad (2.16)$$

where $A^{(m)}$ denotes the contribution of m-loop order,

$$A^{(3)}(u,\epsilon) = -\frac{4}{3}\,n(n+2)\left[\frac{3}{\epsilon}-\frac{40}{\epsilon^2}+\frac{24(n+4)}{\epsilon^3}\right]u^2\,, \qquad (2.17)$$

$$A^{(4)}(u,\epsilon) = -\frac{8}{3}\,n(n+2)\left[\frac{(n+8)(12\zeta(3)-25)}{\epsilon}+\frac{96n+696}{\epsilon^2}\right.$$
$$\left.-\frac{248n+1024}{\epsilon^3}+\frac{48(n+4)(n+5)}{\epsilon^4}\right]u^3\,, \qquad (2.18)$$

$$A^{(5)}(u,\epsilon) = -\frac{2}{15}\,n(n+2)\left[\frac{768(n+4)(n+5)(5n+28)}{\epsilon^5}-\frac{128(293n^2+2624n+5840)}{\epsilon^4}\right.$$

$$
\begin{aligned}
&+\frac{9216\zeta(3)(5n+22)+32(519n^2+8462n+25048)}{\epsilon^3}\\
&-\frac{192\zeta(3)(7n^2-28n+48)+4608\zeta(4)(5n+22)+64(31n^2+2354n+9306)}{\epsilon^2}\\
&+\Big(48\zeta(3)(3n^2-382n-1700)+288\zeta(4)(4n^2+39n+146)\\
&-3072\zeta(5)(5n+22)-3(319n^2-13968n-64864)\Big)\frac{1}{\epsilon}\Big]\,u^4\,, \qquad (2.19)
\end{aligned}
$$

where $\zeta(s)=\sum_{j=1}^{\infty} j^{-s}$ is the Riemann zeta function with $\zeta(3)=1.20205690$, $\zeta(4)=\pi^4/90$ and $\zeta(5)=1.03692776$. Most important is the d-independent RG function $B(u)$ which is determined by [9]

$$
4B(u)=\left[2\zeta_r(u)-\epsilon\right]A(u,\epsilon)+\beta_u(u,\epsilon)\frac{\partial A(u,\epsilon)}{\partial u}\,. \qquad (2.20)
$$

Using $A(u,\epsilon)$ of Eqs. (2.16)–(2.19) and the perturbative expressions for ζ_r and β_u of Refs. [17–19] we find

$$
\begin{aligned}
B(u)=&\frac{n}{2}+3n(n+2)u^2-\frac{8}{3}n(n+2)(n+8)(25-12\zeta(3))u^3\\
&+\frac{1}{2}n(n+2)\Big[16\zeta(3)(3n^2-382n-1700)-1024\zeta(5)(5n+22)\\
&+96\zeta(4)(4n^2+39n+146)-319n^2+13968n+64864\Big]u^4\\
&+O(u^5). \qquad (2.21)
\end{aligned}
$$

In Table II the coefficients $c_{_{Bm}}$ of the power series

$$
B(u)=\sum_{m=0}^{\infty} c_{_{Bm}}u^m \qquad (2.22)
$$

are given for $n=0,1,2,3$ up to $m=4$ corresponding to five-loop order. Table II also contains the coefficients $f_i^{(m)}$ of the power series of the functions

$$
f_i(u)=\sum_{m=1}^{\infty} f_i^{(m)}u^m \qquad (2.23)
$$

where $f_i(u)$ denotes the functions $\tilde{\beta}(u)$, $\zeta_r(u)$ and $\zeta_\phi(u)$ for $i=1,2,3$, respectively. These coefficients are taken from Refs. [17–19]. Up to four-loop order they agree with those in Table 1 of Ref. [8]. (Note that $f_i^{(k)}$ in the table caption of Ref. [8] should read $f_i^{(k)}\times 10^{-4}$.)

The five-loop coefficients $f_1^{(6)}$, $f_2^{(5)}$ and $f_3^{(5)}$ differ from those in Table 1 of Ref. [8] according to the corrections in five-loop order in Ref. [19].

In Fig. 2 the different partial sums of $B(u)$ from two- to five-loop order are shown for the example $n = 2$. As expected, the contributions for $m \geq 2$ have alternating signs and increase considerably in magnitude. Clearly a resummation of $B(u)$ is necessary similar to that for $\zeta_r(u)$, $\zeta_\phi(u)$ and $\tilde{\beta}(u)$ performed previously [8].

First we reexamine the fixed point values $u^\star$, $\beta_u(u^\star, 1) = 0$, for $n = 1, 2, 3$ obtained in Refs. [8,24] by means of Borel resummation on the basis of previous five-loop results [17,18] and in Ref. [16] on the basis of four-loop results. Here we employ the corrected five-loop coefficients for the ϵ expansion of the fixed point value which we have derived from Eq. (8) of Ref. [19] (see Table III). Employing the standard Borel resummation method [8,31] we have obtained the fixed point values in three dimensions

$$u^\star = 0.0404 \quad \pm 0.0003 \qquad \text{for } n = 1\,, \tag{2.24}$$

$$u^\star = 0.0362 \quad \pm 0.0002 \qquad \text{for } n = 2\,, \tag{2.25}$$

$$u^\star = 0.0327 \quad \pm 0.0001 \qquad \text{for } n = 3\,. \tag{2.26}$$

The corresponding resummation parameters α and $b = 5.5 + n/2$ [31] are

$$2.22 \leq \alpha \leq 3.41\,, \quad b = 6.0 \qquad \text{for } n = 1\,, \tag{2.27}$$

$$2.45 \leq \alpha \leq 3.43\,, \quad b = 6.5 \qquad \text{for } n = 2\,, \tag{2.28}$$

$$2.71 \leq \alpha \leq 3.43\,, \quad b = 7.0 \qquad \text{for } n = 3\,. \tag{2.29}$$

The previous fixed point values [8,16,24] are consistent with Eqs. (2.24)–(2.26) within the previous error bars. The present error bars are smaller than the previous ones [8,16,24]. (The range of α determines our error bars, as described further below.)

We have performed Borel resummations of $B(u)$ at the fixed point $u^\star$ for the cases $n = 1$, 2, 3. In addition, for the important case $n = 2$ (superfluid ^{4}He), we have determined the Borel resummed function $B(u)$ at various values of u. The results are given in Eq. (2.34)–(2.36) and in Figs. 2 and 3.

A description of the Borel resummation method [31] for the present purpose has been given in Section 5 of Ref. [8]. (In Eq. (5.10) of Ref. [8] a_{jk} should read $a_{j-m,k}$.) In the present work, however, we use a different way of determining the parameters α and b of the summations. This implies a different determination of the error bars.

For $B(u)$ the value of the parameter b is not known from an analysis of the large-order behavior (see Eq. (5.6) of Ref. [8] and references therein). Here we fix both b and α by requiring fastest convergence of the series of the partial Borel sums $S^{(L)} = S^{(L)}(u, \alpha, b)$ for $B(u)$ defined in Eq. (5.12) of Ref. [8] (here L corresponds to $(L+1)$-loop order). To do so we look for the minima of $\Delta^{(4)}$ and $\Delta^{(3)}$ with regard to variations of both b and α where

$$\Delta^{(L)}(u, \alpha, b) = \left| \frac{S^{(L)} - S^{(L-1)}}{S^{(L-1)}} \right| . \tag{2.30}$$

This yields five-loop values of the parameters α, b for each u. In order to define an error bar, we apply the same method to the four-loop result of $B(u)$, i.e. to $\Delta^{(3)}$ and $\Delta^{(2)}$. The four-loop values of α, b together with the five-loop values provide the ranges of the best values of α and b as a result of the combined four- and five-loop analysis. Then we define the error bar of the five-loop result for $B(u)$ by the maximum and minimum of the resummed four- and five-loop values for $B(u)$ over the ranges of the best values of α, b. At the fixed point $u^\star$, Eqs. (2.24)–(2.26), we find the ranges

$$0.95 \leq \alpha \leq 1.08\,, \quad 5.7 \leq b \leq 7.75 \quad \text{for } n = 1\,, \tag{2.31}$$

$$0.94 \leq \alpha \leq 1.04\,, \quad 7.0 \leq b \leq 8.59 \quad \text{for } n = 2\,, \tag{2.32}$$

$$0.94 \leq \alpha \leq 1.02\,, \quad 8.18 \leq b \leq 9.76 \quad \text{for } n = 3\,. \tag{2.33}$$

The corresponding Borel resummed results for $B(u^\star)$ are

$$B(u^\star) = 0.5024 \quad \pm 0.0011 \qquad \text{for } n = 1\,, \tag{2.34}$$

$$B(u^\star) = 1.0053 \quad \pm 0.0022 \qquad \text{for } n = 2\,, \tag{2.35}$$

$$B(u^\star) = 1.5080 \quad \pm 0.0034 \qquad \text{for } n = 3\,. \tag{2.36}$$

We have also determined the function $B(u)$ for $n = 2$ (superfluid ^{4}He) in the range $0 \leq u \leq 0.04$ as shown in Fig. 3.

Most remarkable is the smallness of the deviation of the resummed function $B(u)$ for $n = 1, 2, 3$ from its two-loop approximation $n/2$. This confirms previous conjectures [9–11] and justifies earlier analyses [7,23,24].

III. AMPLITUDE FUNCTIONS $F_\pm$ IN THREE DIMENSIONS

By means of the renormalized vertex functions in Eq. (2.7) we define the dimensionless amplitude functions

$$F_\pm(\mu\xi_\pm, u, d) = -4\mu^\epsilon A_d^{-1}\Gamma_\pm^{(2,0)}(\xi_\pm, u, \mu, d)\,. \tag{3.1}$$

They enter the critical behavior of the specific heat in three dimensions in the form of the functions

$$F_\pm(1, u, 3) \equiv F_\pm(u) \tag{3.2}$$

according to [9]

$$\mathring{C}^\pm = C_B + \frac{1}{4}a^2\mu^{-1}A_3K_\pm(u(l_\pm))\exp\int_u^{u(l_\pm)}\frac{2\zeta_r(u')-1}{\beta_u(u',1)}\mathrm{d}u' \tag{3.3}$$

where

$$K_\pm(u) = F_\pm(u) - A(u, 1) \tag{3.4}$$

and $a = Z_r(u, 1)^{-1}a_0$. In Eq. (3.3), $u(l)$ is the effective coupling satisfying

$$l\,\frac{\mathrm{d}u(l)}{\mathrm{d}l} = \beta_u(u(l), 1) \tag{3.5}$$

with $u(1) = u$. The flow parameters l_+ and l_- are chosen as $l_+ = (\mu\xi_+)^{-1}$ and $l_- = (\mu\xi_-)^{-1}$ above and below T_c.

The amplitude functions are expandable in integer powers of u [9] and have the power series [10]

$$F_+(u) = \sum_{m=0}^{\infty} c_{Fm}^+ u^m \tag{3.6}$$

and [11]

$$F_-(u) = \frac{1}{u} \sum_{m=0}^{\infty} c^-_{Fm} u^m \, . \tag{3.7}$$

We have determined c^+_{Fm} up to five-loop order (i.e., up to $m = 4$) for $n = 1, 2, 3$ and c^-_{Fm} up to five-loop order (i.e., up to $m = 5$) for $n = 1$ in two different ways.

(i) The coefficients c^+_{Fm} and c^-_{Fm} can be calculated from Eqs. (3.1), (2.7) in three dimensions according to

$$F_\pm(u) = -16\pi Z_r^2 \xi_\pm^{-1} \mathring{\Gamma}^{2,0}_\pm(\xi_\pm, 4\pi\xi_\pm^{-1} Z_u Z_\phi^{-2} u, 3) + A(u, 1), \tag{3.8}$$

where the Z-factors have the arguments $Z_i(u, 1)$. The perturbative expression for $\mathring{\Gamma}^{(2,0)}_+$ can be obtained for $n = 1$, 2, 3 from

$$\mathring{\Gamma}^{(2,0)}_+(\xi_+, u_0, 3) = \frac{1}{4u_0} Z_5^{-1}(\lambda) \tag{3.9}$$

where the renormalization factor $Z_5(\lambda)$ and its relation to the specific heat have been presented by Bervillier and Godrèche [28] and by Bagnuls and Bervillier [29,32], see also Ref. [10]. For $d = 3$ their renormalized coupling λ is related to our u_0 via the renormalization factor $Z_3(\lambda)$ according to $u_0\xi_+ = -2\pi\lambda Z_3(\lambda)^{-1/2}$ as noted in Ref. [10]. The perturbative expression of $\mathring{\Gamma}^{(2,0)}_-$ for $n = 1$ can be determined according to

$$\begin{aligned}\mathring{\Gamma}^{(2,0)}_-(\xi_-, u_0, 3) &= \frac{\partial^2}{\partial r_0'^2} \mathring{\Gamma}_-(\xi_-, u_0, 3) \\ &= \left(\frac{\partial r_0'}{\partial \xi_-}\right)^{-1} \frac{\partial}{\partial \xi_-} \left[\left(\frac{\partial r_0'}{\partial \xi_-}\right)^{-1} \frac{\partial}{\partial \xi_-} \mathring{\Gamma}_-(\xi_-, u_0, 3) \right],\end{aligned} \tag{3.10}$$

where $r_0'(\xi_-, u_0)$ is given by Eq. (3.8) of Ref. [11]. The Helmholtz free energy $\mathring{\Gamma}_-(\xi_-, u_0, 3)$ is given by Eq. (3.15) of Ref. [11] where our $\mathring{\Gamma}_-(\xi_-, u_0, 3)$ is denoted by $\tilde{\Gamma}_{-0}(\xi_-, u_0)$. Our numerical results for $c^\pm_{Fm}$ up to nine digits are presented in Table IV.

(ii) Alternatively the coefficients $c^\pm_{Fm}$ can be determined via the relation

$$8A_3^{-1} P_\pm(u) f^{(3,0)}_\pm(u) = (1 - 2\zeta_r(u)) F_\pm(u) + 4B(u) - \beta_u(u) \partial F_\pm(u)/\partial u \tag{3.11}$$

as done previously [10,11]. For the definition of $P_\pm$ and $f^{(3,0)}_\pm$ and for a derivation of Eq. (3.11) we refer to Refs. [8] and [9]. In the present context we need the contributions to $P_\pm$

and $f_\pm^{(3,0)}$ only up to $O(u^4)$ as given in Table 4 of Ref. [10] and Table 3 of Ref. [11] since $B(u)$ is known only up to $O(u^4)$ as well. (We recall that the coefficients of P_- are determined by those of P_+ according to $P_-(u) = -\frac{1}{2}\{1 + 2[1 - P_+(u)] - \frac{3}{2}\zeta_r(u)\}$ [9].) This calculation via Eq. (3.11) yields coefficients $c_{Fm}^{\pm}$ that agree with those obtained via Eq. (3.8) up to eight digits for c_{Fm}^{-} and up to seven digits for c_{Fm}^{+}. We consider the calculation (i) via Eq. (3.8) as slightly more reliable since fewer numerical operations are required than in the calculation (ii) using Eq. (3.11).

Since here we have used the perturbative contributions of $B(u)$ up to five-loop order, the resulting higher-order coefficients $c_{Fm}^{\pm}$ given in Table IV differ from those determined previously (see Table 4 of Ref. [10] and Table 3 of Ref. [11]) where the approximation $B(u) = n/2 + O(u^2)$ was used. Only our low-order coefficients c_{F0}^{+}, c_{F1}^{+}, c_{F0}^{-}, c_{F1}^{-} and c_{F2}^{-} agree with the previous ones [10,11]. The coefficients c_{Fm}^{+} with $m > 1$ differ considerably from the previous ones whereas the coefficients c_{Fm}^{-} with $m > 2$ differ only by 0.2% ($m = 3$), 0.1% ($m = 4$), and 2% ($m = 5$).

In order to study the effect of the new higher-order terms we have performed Borel resummations of the series for $uF_-(u)$ and for $u[F_-(u) - F_+(u)]$ at the fixed point $u^\star$ for the case $n = 1$. The method employed is the same as for $B(u)$ in Sect. II. The parameter ranges turn out to be

$$1.6 \leq \alpha \leq 1.7\,, \quad 7.48 \leq b \leq 8.70 \tag{3.12}$$

for $u^\star F_-(u^\star)$, and

$$1.4 \leq \alpha \leq 1.7\,, \quad 6.0 \leq b \leq 11.7 \tag{3.13}$$

for $u^\star[F_-(u^\star) - F_+(u^\star)]$.

We have found that our present method does not yield a reliable estimate of the parameters α and b for $F_+(u^\star)$ separately; this may be related to the fact that, unlike c_{Fm}^{-}, the coefficients c_{Fm}^{+} (see Table IV) do not have alternating signs for $m \leq 3$ (this alternation is predicted for the asymptotic large-order behavior [8,31]). $F_+(u)$ will be further studied

elsewhere. In the application to amplitude ratios given below we shall not need $F_+(u^\star)$ separately.

The resummation results are

$$u^\star F_-(u^\star) = 0.3687 \pm 0.0040 \tag{3.14}$$

and

$$u^\star[F_-(u^\star) - F_+(u^\star)] = 0.4170 \pm 0.0036\,. \tag{3.15}$$

The previous approximate resummation results [10,11] for $n = 1$ were $u^\star F_-(u^\star) = 0.3648$ and $u^\star[F_-(u^\star) - F_+(u^\star)] = 0.4170$ with an error bar of about 1%. Thus our resummation results differ from the previous ones only by about 1% for F_- and by less than 0.1% for $F_- - F_+$, confirming previous conjectures [10,11]. The parameter d_F in the effective representation [11] $F_-(u) = (2u)^{-1} - 4(1 + d_F u)$ now becomes $d_F = -4.64$ (compared to -4.04 in Ref. [11]). This leaves the solid line in Fig. 4 of Ref. [11] essentially unchanged.

As an illustration we apply our results to the specific heat in the asymptotic critical region where it can be represented as [1,9]

$$\mathring{C}^{\pm} = \frac{A^{\pm}}{\alpha}|t|^{-\alpha}\left(1 + a_c^{\pm}|t|^{\Delta} + \cdots\right) + B \tag{3.16}$$

with the Wegner exponent [33] Δ. We consider the universal amplitude ratios [1] A^+/A^- and a_c^+/a_c^-. The former can be expressed in terms of $B^\star \equiv B(u^\star)$ and $F_\pm^\star \equiv F_\pm(u^\star)$ in three dimensions as [9]

$$\frac{A^+}{A^-} = \left(\frac{b_+}{b_-}\right)^{\alpha}\left[1 + \alpha\frac{F_-^\star - F_+^\star}{4\nu B^\star + \alpha F_-^\star}\right] \tag{3.17}$$

where

$$\frac{b_+}{b_-} = \frac{2\nu P_+^\star}{(3/2) - 2\nu P_+^\star}. \tag{3.18}$$

with $P_+^\star \equiv P_+(u^\star)$. For the Borel resummed value of $P_+^\star$ we have obtained 0.757 ± 0.004 for $n = 1$. The expression of a_c^+/a_c^- is more complicated and depends on the derivatives of the

functions F_-, $F_- - F_+$, P_+, B, and ζ_r at the fixed point, as specified in Eqs. (4.24), (5.19) and (5.21)–(5.24) of Ref. [9]. We have performed Borel resummations for these quantities on the basis of our new results. Due to the present lack of knowledge on higher-order terms of F_- below T_c for $n > 1$ we have confined ourselves to the case $n = 1$.

Our result for A^+/A^- depends on the value employed for the critical exponent $\nu = (2-\alpha)/3$. For $\nu = 0.6310$ [13] we obtain $A^+/A^- = 0.539$, with an uncertainty of about 2%. For a_c^+/a_c^- we obtain the value 1.0, with an uncertainty of $O(10\%)$. A more detailed presentation of these applications including error bars and a comparison with previous results will be given elsewhere. Here we only note that $B^\star$ and $F_\pm^\star$ enter also various other important universal ratios, e.g., R_ξ^T and a_c^-/a_{ρ_s} related to the superfluid density ρ_s [9]. A quantitative calculation of these ratios must be postponed until higher-order results are available for $n > 1$ below T_c.

After completion of the present work we learned of the preprint hep-ph/9710346 by B. Kastening, "Five-Loop Vacuum Energy Function in ϕ^4 Theory with O(n)-Symmetric and Cubic Interactions" where the perturbative terms of a function equivalent to $B(u)$ have been calculated up to five-loop order. These terms agree with ours in Eq. (2.21).

ACKNOWLEDGEMENTS

We gratefully acknowledge support by Deutsches Zentrum für Luft- und Raumfahrt (DLR, previously DARA) under grant number 50 WM 9669 as well as by NASA under contract number 960838.

APPENDIX A: DERIVATION OF THE ADDITIVE RENORMALIZATION $A(u,\epsilon)$

The following expressions (A1)–(A54) give the ultraviolet $d = 4$ poles of the diagrams shown in Fig. 1 defined by $KR'(\partial^2 I^{(i)}/\partial r_0^2)$ according to the standard notations, see e.g. Ref. [18]. Here R' denotes the incomplete ultraviolet R-operation which subtracts subdivergences, and K denotes the operation of taking pole parts. We use subscripts $(a.b)$ for the pole terms on the right-hand-sides of (A1)–(A8) that are identical with the numbers associated with the three- and four-loop diagrams of Ref. [16] (the first number a in the brackets indicates the number of loops and the second number b indicates the consecutive number of a diagram in Table 1 of Ref. [16]). The subscripts on the right-hand sides of Eqs. (A9)–(A18) correspond to the numbers of the five-loop diagrams of Ref. [18]. We have multiplied the left-hand sides by factors $(16\pi^2)^m$ in m-loop order because of the definition of the bare four-point coupling $16\pi^2 g_0/24$ in Refs. [16–19].

The one- and two-loop expressions read

$$16\pi^2 KR'\left(\frac{\partial^2}{\partial r_0^2}\left[-\frac{1}{2}\int_p \ln(r_0+p^2)\right]\right) = \frac{1}{2}\, I_{(1.1)} \equiv J^{(1)}\,, \tag{A1}$$

$$(16\pi^2)^2 KR'\left(\frac{\partial^2}{\partial r_0^2}\, I^{(2)}(r_0,\epsilon)\right) = 2I_{(2.2)} \equiv J^{(2)}\,, \tag{A2}$$

the three-loop expressions read

$$(16\pi^2)^3 KR'\left(\frac{\partial^2}{\partial r_0^2}\, I^{(3)}(r_0,\epsilon)\right) = 2I_{(3.2)} \equiv J^{(3)}\,, \tag{A3}$$

$$(16\pi^2)^3 KR'\left(\frac{\partial^2}{\partial r_0^2}\, I^{(4)}(r_0,\epsilon)\right) = 8I_{(3.4)} + 12I_{(3.9)} \equiv J^{(4)}\,, \tag{A4}$$

the four-loop expressions read

$$(16\pi^2)^4 KR'\left(\frac{\partial^2}{\partial r_0^2}\, I^{(5)}(r_0,\epsilon)\right) = 2I_{(4.5)} \equiv J^{(5)}\,, \tag{A5}$$

$$(16\pi^2)^4 KR'\left(\frac{\partial^2}{\partial r_0^2}\, I^{(6)}(r_0,\epsilon)\right) = 0 \equiv J^{(6)}\,, \tag{A6}$$

$$(16\pi^2)^4 KR'\left(\frac{\partial^2}{\partial r_0^2}\, I^{(7)}(r_0,\epsilon)\right) = 4I_{(4.9)} + 6I_{(4.11)} \equiv J^{(7)}\,, \tag{A7}$$

$$(16\pi^2)^4 KR'\left(\frac{\partial^2}{\partial r_0^2}\, I^{(8)}(r_0,\epsilon)\right) = 24I_{(4.12)} + 6I_{(4.18)} + 12I_{(4.19)} \equiv J^{(8)}\,, \tag{A8}$$

and the five-loop expressions read

$$(16\pi^2)^5 KR'\left(\frac{\partial^2}{\partial r_0^2}\, I^{(9)}(r_0,\epsilon)\right) = 2I_{117} \equiv J^{(9)}\,, \tag{A9}$$

$$(16\pi^2)^5 KR'\left(\frac{\partial^2}{\partial r_0^2}\, I^{(10)}(r_0,\epsilon)\right) = 0 \equiv J^{(10)}\,, \tag{A10}$$

$$(16\pi^2)^5 KR'\left(\frac{\partial^2}{\partial r_0^2}\, I^{(11)}(r_0,\epsilon)\right) = 0 \equiv J^{(11)}\,, \tag{A11}$$

$$(16\pi^2)^5 KR'\left(\frac{\partial^2}{\partial r_0^2}\, I^{(12)}(r_0,\epsilon)\right) = 2I_{121} \equiv J^{(12)}\,, \tag{A12}$$

$$(16\pi^2)^5 KR'\left(\frac{\partial^2}{\partial r_0^2}\, I^{(13)}(r_0,\epsilon)\right) = 4I_7 + 6I_{120} \equiv J^{(13)}\,, \tag{A13}$$

$$(16\pi^2)^5 KR'\left(\frac{\partial^2}{\partial r_0^2}\, I^{(14)}(r_0,\epsilon)\right) = 2I_8 \equiv J^{(14)}\,, \tag{A14}$$

$$(16\pi^2)^5 KR'\left(\frac{\partial^2}{\partial r_0^2}\, I^{(15)}(r_0,\epsilon)\right) = 2I_{14} + 12I_{15} + 12I_{19} + 24I_{21} + 4I_{23} + 18I_{106} \equiv J^{(15)}\,, \tag{A15}$$

$$(16\pi^2)^5 KR'\left(\frac{\partial^2}{\partial r_0^2}\, I^{(16)}(r_0,\epsilon)\right) = 8I_{79} + 16I_{88} + 32I_{95} + 16I_{100} \equiv J^{(16)}\,, \tag{A16}$$

$$(16\pi^2)^5 KR'\left(\frac{\partial^2}{\partial r_0^2}\, I^{(17)}(r_0,\epsilon)\right) = 2I_{81} + 8I_{84} + 4I_{99} \equiv J^{(17)}\,, \tag{A17}$$

$$(16\pi^2)^5 KR'\left(\frac{\partial^2}{\partial r_0^2}\, I^{(18)}(r_0,\epsilon)\right) = 8I_{32} + 32I_{47} + 4I_{73} + 8I_{93} + 8I_{98} + 4I_{109} + 8I_{116} \equiv J^{(18)}\,. \tag{A18}$$

The pole terms up to four loops are [16]

$$I_{(1.1)} = \frac{2}{\epsilon}\,, \tag{A19}$$

$$I_{(2.2)} = -\frac{4}{\epsilon^2}\,, \tag{A20}$$

$$I_{(3.2)} = \frac{8}{\epsilon^3}\,, \tag{A21}$$

$$I_{(3.4)} = \frac{2}{3\epsilon^2} - \frac{3}{4\epsilon}\,, \tag{A22}$$

$$I_{(3.9)} = \frac{8}{3\epsilon^3} - \frac{8}{3\epsilon^2} + \frac{2}{3\epsilon}\,, \tag{A23}$$

$$I_{(4.5)} = -\frac{16}{\epsilon^4}\,, \tag{A24}$$

$$I_{(4.9)} = -\frac{4}{3\epsilon^3} + \frac{3}{2\epsilon^2}\,, \tag{A25}$$

$$I_{(4.11)} = -\frac{16}{3\epsilon^4} + \frac{16}{3\epsilon^3} - \frac{4}{3\epsilon^2}\,, \tag{A26}$$

$$I_{(4.12)} = -\frac{4}{3\epsilon^4} + \frac{10}{3\epsilon^3} - \frac{13}{3\epsilon^2} + \frac{11 - 6\zeta(3)}{6\epsilon}\,, \tag{A27}$$

$$I_{(4.18)} = -\frac{8}{3\epsilon^4} + \frac{8}{3\epsilon^3} + \frac{4}{3\epsilon^2} + \frac{2\zeta(3) - 2}{\epsilon}\,, \tag{A28}$$

$$I_{(4.19)} = -\frac{2}{3\epsilon^3} + \frac{1}{\epsilon^2} - \frac{7}{12\epsilon}\,, \tag{A29}$$

and the five-loop pole terms are [18]

$$I_7 = \frac{8}{3\epsilon^4} - \frac{3}{\epsilon^3}\,, \tag{A30}$$

$$I_8 = \frac{8}{3\epsilon^4} - \frac{3}{\epsilon^3}\,, \tag{A31}$$

$$I_{14} = \frac{4}{15\epsilon^3} - \frac{3}{10\epsilon^2} - \frac{5}{96\epsilon}\,, \tag{A32}$$

$$I_{15} = \frac{2}{15\epsilon^3} - \frac{13}{20\epsilon^2} + \frac{857}{960\epsilon}\,, \tag{A33}$$

$$I_{19} = \frac{8}{15\epsilon^4} - \frac{7}{3\epsilon^3} + \frac{25}{6\epsilon^2} - \frac{215}{96\epsilon}\,, \tag{A34}$$

$$I_{21} = \frac{16}{15\epsilon^4} - \frac{7}{5\epsilon^3} - \frac{11}{60\epsilon^2} + \frac{157}{320\epsilon}\,, \tag{A35}$$

$$I_{23} = \frac{4}{15\epsilon^3} - \frac{3}{10\epsilon^2} - \frac{5}{96\epsilon}\,, \tag{A36}$$

$$I_{32} = \frac{48\zeta(3)}{5\epsilon^3} - \frac{48\zeta(3) + 24\zeta(4)}{5\epsilon^2} + \frac{14\zeta(3) + 9\zeta(4) - 16\zeta(5)}{5\epsilon}\,, \tag{A37}$$

$$I_{47} = \frac{8}{15\epsilon^5} - \frac{12}{5\epsilon^4} + \frac{6}{\epsilon^3} + \frac{18\zeta(3) - 45}{5\epsilon^2} + \frac{146 - 90\zeta(3) - 9\zeta(4)}{30\epsilon}\,, \tag{A38}$$

$$I_{73} = \frac{16}{15\epsilon^5} - \frac{8}{3\epsilon^4} + \frac{28}{15\epsilon^3} + \frac{6 + 4\zeta(3)}{5\epsilon^2} - \frac{32 - 12\zeta(3) - 18\zeta(4)}{30\epsilon}\,, \tag{A39}$$

$$I_{79} = \frac{16}{5\epsilon^5} - \frac{16}{5\epsilon^4} - \frac{8}{5\epsilon^3} + \frac{4 + 4\zeta(3)}{5\epsilon^2} + \frac{7 + 6\zeta(3) - 12\zeta(4)}{5\epsilon}\,, \tag{A40}$$

$$I_{81} = \frac{16}{3\epsilon^5} - \frac{16}{3\epsilon^4} - \frac{8}{3\epsilon^3} + \frac{4 - 4\zeta(3)}{\epsilon^2}\,, \tag{A41}$$

$$I_{84} = \frac{8}{3\epsilon^5} - \frac{20}{3\epsilon^4} + \frac{26}{3\epsilon^3} - \frac{44 - 24\zeta(3)}{12\epsilon^2}\,, \tag{A42}$$

$$I_{88} = \frac{16}{15\epsilon^5} - \frac{8}{3\epsilon^4} + \frac{28}{15\epsilon^3} + \frac{6 - 12\zeta(3)}{5\epsilon^2} - \frac{16 - 18\zeta(4)}{15\epsilon}\,, \tag{A43}$$

$$I_{93} = \frac{8}{5\epsilon^5} - \frac{52}{15\epsilon^4} + \frac{34}{15\epsilon^3} + \frac{116 - 24\zeta(3)}{60\epsilon^2} - \frac{56 - 44\zeta(3) + 6\zeta(4)}{20\epsilon}\,, \tag{A44}$$

$$I_{95} = \frac{16}{15\epsilon^5} - \frac{8}{3\epsilon^4} + \frac{28}{15\epsilon^3} + \frac{6 - 12\zeta(3)}{5\epsilon^2} - \frac{16 - 18\zeta(4)}{15\epsilon}\,, \tag{A45}$$

$$I_{98} = \frac{4}{15\epsilon^4} - \frac{14}{15\epsilon^3} + \frac{19}{15\epsilon^2} - \frac{386 + 768\zeta(3)}{960\epsilon}\,, \tag{A46}$$

$$I_{99} = \frac{4}{3\epsilon^4} - \frac{2}{\epsilon^3} + \frac{7}{6\epsilon^2}\,, \tag{A47}$$

$$I_{100} = \frac{4}{5\epsilon^4} - \frac{6}{5\epsilon^3} + \frac{1}{5\epsilon^2} + \frac{81 - 48\zeta(3)}{160\epsilon}\,, \tag{A48}$$

$$I_{106} = \frac{64}{15\epsilon^5} - \frac{32}{5\epsilon^4} + \frac{8}{5\epsilon^3} + \frac{16}{15\epsilon^2} - \frac{2}{15\epsilon}\,, \tag{A49}$$

$$I_{109} = \frac{16}{15\epsilon^5} - \frac{16}{5\epsilon^4} + \frac{16}{5\epsilon^3} + \frac{52-108\zeta(3)}{15\epsilon^2} - \frac{202-168\zeta(3)-18\zeta(4)}{30\epsilon}\,, \tag{A50}$$

$$I_{116} = \frac{8}{15\epsilon^4} - \frac{4}{3\epsilon^3} + \frac{32}{15\epsilon^2} - \frac{250-96\zeta(3)}{120\epsilon}\,, \tag{A51}$$

$$I_{117} = \frac{32}{\epsilon^5}\,, \tag{A52}$$

$$I_{120} = \frac{32}{3\epsilon^5} - \frac{32}{3\epsilon^4} + \frac{8}{3\epsilon^3}\,, \tag{A53}$$

$$I_{121} = \frac{32}{3\epsilon^5} - \frac{32}{3\epsilon^4} + \frac{8}{3\epsilon^3}\,. \tag{A54}$$

In Eqs. (A37) and (A51) the corrections found in Ref. [19] have been taken into account. Note that in Eqs. (A19)–(A54) we have used $\epsilon = 4-d$ whereas in Refs. [16,18] ϵ denotes $(4-d)/2$.

Eqs. (A1)–(A54) determine the additive renormalization according to Eq. (2.7) as

$$A(u,\epsilon) = -2\left[\frac{n}{\epsilon} + \frac{8n(n+2)}{\epsilon^2}(-u/2) + \sum_{i=3}^{4} S^{(i)}G^{(i)}J^{(i)}(-u/2)^2 \right.$$
$$\left. + \sum_{i=5}^{8} S^{(i)}G^{(i)}J^{(i)}(-u/2)^3 + \sum_{i=9}^{18} S^{(i)}G^{(i)}J^{(i)}(-u/2)^4\right]. \tag{A55}$$

The overall factor of 2 in Eq. (A55) arises from the $d=4$ value of the factor $(A_d/4)^{-1}/16\pi^2$ which is needed to obtain $A(u,\epsilon)$ from Eqs. (A1)–(A54) according to Eq. (2.7). The renormalized coupling u enters Eq. (A55) in the form $u/2$; the factor 1/2 takes into account that, near $d=4$, $u_0 = A_d^{-1}u + O(u^2) = 8\pi^2 u + O(u^2) \neq 16\pi^2 u + O(u^2)$ [see Eq. (2.8)].

APPENDIX B: Z FACTORS

In deriving the coefficients of the perturbation series of the quantities $F_\pm(u)$, $P_\pm(u)$, and $f_\pm^{(3,0)}(u)$ we needed the Z-factors Z_r, Z_ϕ, and Z_u calculated previously [17–19] up to five-loop order. Since their explicit form is not available in the literature we present them here explicitly. They read

$$Z_r(u,\epsilon) = 1 + \sum_{m=1}^{5} Z_r^{(m)}(\epsilon)u^m + O(u^6)\,, \tag{B1}$$

$$Z_u(u,\epsilon) = 1 + \sum_{m=1}^{5} Z_u^{(m)}(\epsilon)u^m + O(u^6)\,, \tag{B2}$$

$$Z_\phi(u,\epsilon) = 1 + \sum_{m=1}^{5} Z_\phi^{(m)}(\epsilon)u^m + O(u^6)\,, \tag{B3}$$

with the following coefficients in m-loop order:

Coefficients of $Z_r(u,\epsilon)$:

$$Z_r^{(1)}(\epsilon) = \frac{4(n+2)}{\epsilon}\,, \tag{B4}$$

$$Z_r^{(2)}(\epsilon) = 4(n+2)\left[\frac{4(n+5)}{\epsilon^2} - \frac{5}{\epsilon}\right], \tag{B5}$$

$$Z_r^{(3)}(\epsilon) = \frac{16}{3}\,(n+2)\left[\frac{15n+111}{\epsilon} + \frac{-278-61n}{\epsilon^2} + \frac{12n^2+132n+360}{\epsilon^3}\right], \tag{B6}$$

$$\begin{aligned} Z_r^{(4)}(\epsilon) = -\frac{2}{3}\,(n+2)\Bigg[&\frac{288\zeta(4)(5n+22)+48\zeta(3)(3n^2+10n+68)+31060-n^2+7578n}{\epsilon} \\ &- \frac{1152\zeta(3)(22+5n)+1236n^2+23580n+74616}{\epsilon^2} + \frac{16(245n^2+2498n+6284)}{\epsilon^3} \\ &-192\,\frac{(n+5)(2n+13)(n+6)}{\epsilon^4}\Bigg], \end{aligned} \tag{B7}$$

$$\begin{aligned} Z_r^{(5)}(\epsilon) = \frac{4}{15}(n+2)\Bigg[&\Big(9600\zeta(6)(55n+2n^2+186)+768\zeta(5)(-5n^2+72+14n) \\ &+288\zeta(4)(29n^2+2668-3n^3+816n)+768\zeta(3)^2(-2n^2-145n-582) \\ &+48\zeta(3)(8208n+17n^3+940n^2+31848)+21n^3+45254n^2+1077120n \\ &+3166528\Big)\frac{1}{\epsilon} - \Big(30720\zeta(5)(2n^2+186+55n)+576\zeta(4)(5n+22)(n-22) \\ &+96\zeta(3)(27n^3+1224n^2+14456n+45448)-98n^3+277280n^2+3073376n \\ &+7449712\Big)\frac{1}{\epsilon^2} + \Big(2304\zeta(3)(13n+74)(5n+22)+21576n^3+685192n^2 \end{aligned}$$

$$+ 5017312n + 10459360\Big)\frac{1}{\epsilon^3} - \frac{32(307976 + 31752n^2 + 172176n + 1933n^3}{\epsilon^4}$$
$$+ \frac{384(5n+34)(n+6)(n+5)(2n+13)}{\epsilon^5}\Bigg]\,. \tag{B8}$$

Coefficients of $Z_\phi(u,\epsilon)$:

$$Z_\phi^{(1)} = 0\,, \tag{B9}$$
$$Z_\phi^{(2)} = -\frac{4(n+2)}{\epsilon}\,, \tag{B10}$$
$$Z_\phi^{(3)} = \frac{8}{3}\,(n+2)(n+8)\left[\frac{1}{\epsilon} - \frac{4}{\epsilon^2}\right], \tag{B11}$$
$$Z_\phi^{(4)} = 2(n+2)\left[\frac{5(n^2-18n-100)}{\epsilon} + \frac{4(n^2+234+53n)}{\epsilon^2} - \frac{16(n+8)^2}{\epsilon^3}\right], \tag{B12}$$
$$Z_\phi^{(5)} = -\frac{8}{15}\,(n+2)\Bigg[\Big(-1152\zeta(4)(5n+22) + 48\zeta(3)(-6n^2+184+n^3+64n)$$
$$- 22752n - 39n^3 - 296n^2 - 77056\Big)\frac{1}{\epsilon}$$
$$+ \frac{2304\zeta(3)(5n+22) - 60n^3 + 135488 + 42440n + 1844n^2}{\epsilon^2}$$
$$- \frac{16(n+8)(3n^2+269n+1210)}{\epsilon^3} + \frac{192(n+8)^3}{\epsilon^4}\Bigg]\,. \tag{B13}$$

Coefficients of $Z_u(u,\epsilon)$:

$$Z_u^{(1)} = \frac{4(n+8)}{\epsilon}\,, \tag{B14}$$
$$Z_u^{(2)} = 16\left[\frac{(n+8)^2}{\epsilon^2} - \frac{5n+22}{\epsilon}\right], \tag{B15}$$
$$Z_u^{(3)} = \frac{8}{3}\Bigg[\frac{96\zeta(3)(5n+22) + 942n + 2992 + 35n^2}{\epsilon} - \frac{16(n+8)(17n+76)}{\epsilon^2}$$
$$+ \frac{24(n+8)^3}{\epsilon^3}\Bigg], \tag{B16}$$
$$Z_u^{(4)} = -\frac{16}{3}\Bigg[\Big(480\zeta(5)(2n^2+55n+186) - 72\zeta(4)(n+8)(5n+22)$$
$$+ 24\zeta(3)(63n^2+764n+2332) + 20624n + 1640n^2 - 5n^3 + 49912\Big)\frac{1}{\epsilon}$$
$$- \frac{480\zeta(3)(5n+22)(n+8) + 67424n + 153088 + 7736n^2 + 172n^3}{\epsilon^2}$$
$$+ \frac{16(55n+248)(n+8)^2}{\epsilon^3} - \frac{48(n+8)^4}{\epsilon^4}\Bigg], \tag{B17}$$
$$Z_u^{(5)} = \frac{4}{15}\Bigg[\Big(6912\zeta(7)(25774 + 9261n + 686n^2) - 28800\zeta(6)(n+8)(2n^2+55n+186)$$

$$
\begin{aligned}
&+768\zeta(5)(165084+7466n^2+305n^3+66986n)-288\zeta(4)(62656+4084n^2+28084n\\
&+189n^3)+2304\zeta(3)^2(3264-59n^2-6n^3+446n)+48\zeta(3)(1264n^3-13n^4+1312864\\
&+551032n+67432n^2)+20429248n+2518864n^2+195n^4+40148480+39230n^3\Big)\frac{1}{\epsilon}\\
&-\Big(99840\zeta(5)(n+8)(2n^2+55n+186)-14976\zeta(4)(5n+22)(n+8)^2\\
&+3456\zeta(3)(91n^3+15436n+34144+2196n^2)+63219712n-800n^4\\
&+420800n^3+117768192+9811712n^2\Big)\frac{1}{\epsilon^2}\\
&+\frac{66048\zeta(3)(5n+22)(n+8)^2+32(n+8)(733n^3+40186n^2+353392n+803328)}{\epsilon^3}\\
&-\frac{512(193n+875)(n+8)^3}{\epsilon^4}+\frac{3840(n+8)^5}{\epsilon^5}\Bigg].
\end{aligned}
\tag{B18}
$$

REFERENCES

[1] V. Privman, P. C. Hohenberg and A. Aharony, in *Phase Transitions and Critical Phenomena*, ed. by C. Domb and J. L. Lebowitz (Academic Press, London, 1991), Vol. 14, p. 1; and references therein.

[2] G. Ahlers, in: *Quantum Liquids*, ed. by J. Ruvalds and T. Regge (North Holland, Amsterdam, 1978), p. 1; A. Singsaas and G. Ahlers, Phys. Rev. **B30**, 5103 (1984).

[3] J. A. Lipa, V. Dohm, U. E. Israelsson, and M.J. DiPirro, NASA Proposal, NRA 94-OLMSA-05 (1995).

[4] M. Larson, F.C. Liu, and U. E. Israelsson, Czech. J. Phys. **46**, Suppl. S1, 179 (1996).

[5] J. A. Lipa, D.R. Swanson, J. A. Nissen, T.C.P. Chui, and U. E. Israelsson, Phys. Rev. Lett. **76**, 944 (1996).

[6] V. Dohm, J. Low Temp. Phys. **69**, 51 (1987).

[7] V. Dohm, Z. Phys. **B 60**, 61 (1985).

[8] R. Schloms and V. Dohm, Nucl. Phys. **B328**, 639 (1989).

[9] R. Schloms and V. Dohm, Phys. Rev. **B42**, 6142 (1990); ibid. B **46**, 5883 (1992) (E).

[10] H. J. Krause, R. Schloms and V. Dohm, Z. Phys. **B79**, 287 (1990); **B80**, 313 (1990) (E).

[11] F. J. Halfkann and V. Dohm, Z. Phys. **B89**, 79 (1992).

[12] S. S. C. Burnett, M. Strösser, and V. Dohm, Nucl. Phys. **B 504** [**FS**], 665 (1997).

[13] J. C. Le Guillou and J. Zinn-Justin, J. Phys. Lett. (Paris) **46**, L-137 (1985); J. Phys. (Paris) **48**, 19 (1987); **50**, 1365 (1989).

[14] G. t'Hooft and M. Veltman, Nucl. Phys. **B44**, 189 (1972); G. t'Hooft, ibid. **61**, 455 (1973).

[15] D. J. Amit, *Field Theory, the Renormalization Group and Critical Phenomena* (Mc Graw-Hill, New York, 1978).

[16] D. I. Kazakov, O. V. Tarasov, and A. A. Vladimirov, Preprint JINR E2-12249 (1979) Dubna; Zh. Eksp. Teor. Fiz. **77**, 1035 (1979) [A. A. Vladimirov, D. I. Kazakov and O. V. Tarasov, Sov. Phys. JETP **50**, 521 (1979)].

[17] K. G. Chetyrkin, S. G. Gorishny, S. A. Larin and F. V. Tkachov, Phys. Lett. **B132**, 351 (1983).

[18] S. G. Gorishny, S. A. Larin, F. V. Tkachov, K. G. Chetyrkin, *Analytical calculation of five-loop approximations to renormalization group functions in the* $g\varphi^4_{(4)}$ *model in the minimal subtraction scheme: diagrammatic analysis*, Report P-0453 of Institute for Nuclear Research (1986), Moscow (in Russian).

[19] H. Kleinert, J. Neu, V. Schulte-Frohlinde, K. G. Chetyrkin and S. A. Larin, Phys. Lett. **B 272**, 39 (1991); **B 319**, 545 (1993) (E).

[20] V. Dohm, in *Proceedings of the 1997 NASA/JPL Workshop on Fundamental Physics in Microgravity*, Santa Barbara, May 7-9, 1997, edited by D. Strayer (JPL, Pasadena, 1997), p. xxx; DRL (DARA) Proposal, 50 WM 9669 (1997).

[21] G. Parisi, in *Proceedings of the 1973 Cargèse Summer Institute* (unpublished); J. Stat. Phys. **23**, 49 (1980).

[22] K. Symanzik, Lett. Nuovo Cimento **8**, 771 (1973).

[23] V. Dohm, Phys. Rev. Lett. **53**, 1379, 2520 (1984); in *Proceedings of the* 7^{th} *International Conference on Low Temperature Physics*, edited by U. Eckern, A. Schmidt, W. Weber, and H. Wühl (North Holland, Amsterdam, 1984), p. 953; in *Applications of Field Theory to Statistical Mechanics*, edited by L. Garrido (Springer, Berlin, 1985), p. 263; R. Schloms, J. Eggers, and V. Dohm, Jpn. J. Appl. Phys. Suppl. **26-3**, 49 (1987).

[24] R. Schloms and V. Dohm, Europhys. Lett. **3**, 413 (1987).

[25] D. Greywall and G. Ahlers, Phys. Rev. Lett. **28**, 1251 (1972); Phys. Rev. A **7**, 2145 (1973).

[26] G. Ahlers, Phys. Rev. **A3**, 696 (1971); **A8**, 530 (1973); K.H. Mueller, G. Ahlers, and F. Pobell, Phys. Rev. **B 14**, 2096 (1976); J. A. Lipa and T. C. P. Chui, Phys. Rev. Lett. **51**, 2291 (1983).

[27] B. G. Nickel, D. I. Meiron, and G. A. Baker, *University of Guelph Report* (1977), unpublished.

[28] C. Bervillier and C. Godrèche, Phys. **B21**, 5427 (1980).

[29] C. Bagnuls and C. Bervillier, Phys. Rev. **B 32**, 7209 (1985).

[30] C. Bagnuls, C. Bervillier, D. I. Meiron, and B. G. Nickel, Phys. Rev. **B 35**, 3585 (1987).

[31] J. Zinn-Justin, Phys. Rep. **70**, 109 (1981).

[32] In the headline of Table I of Ref. [29], Z_5^{-1}/λ has to be replaced by Z_5^{-1}.

[33] F. Wegner, Phys. Rev. B **5**, 4529 (1972).

[34] A. A. Vladimirov, Teor. Mat. Fiz. **43**, 210 (1980).

FIGURES

FIG. 1. Vacuum diagrams up to five-loop order determining the Helmholtz free energy Γ_0 for $M_0 = 0$, $r_0 > 0$. Diagrams (6), (10) and (11) do not contribute to $A(u, \epsilon)$. The pole terms derived from the vacuum diagrams are given in (A1)–(A54) of App. A.

FIG. 2. Partial sums $B_M(u) = \sum_{m=0}^{M} c_{Bm} u^m$ of $B(u)$, Eq. (2.22), as a function of u for $n = 2$ from $M = 1$ (two-loop order) to $M = 4$ (five-loop order). Also shown is the Borel resummed result (solid line) which deviates from the two-loop result $B_1 = 1$ by only 0.5 % at the fixed point $u^\star = 0.0362$.

FIG. 3. Borel resummation result for the function $B(u)$, Eq. (2.21), for $n = 2$ (solid line) obtained by interpolation between the resummed values of $B(u)$ at $u_k = k\, u^\star/10$, $k = 1, \ldots, 10$ of the renormalized coupling u in the range $0 < u \leq u^\star = 0.0362$, with error bars. Also shown is the three-loop result $B_2(u) = 1 + 24u^2$ (dashed line), compare Fig. 2. The two-loop result is $B_1 = 1$. The Borel values $B_k \equiv B(u_k)$ are 1.000233, 1.00073, 1.0013, 1.0019, 1.0026, 1.0032, 1.0037, 1.0043, 1.0048, 1.0053 for $k = 1, \ldots, 10$, respectively.

TABLES

TABLE I. Symmetry and group factors of the vacuum diagrams shown in Fig. 1. Diagrams (6), (10) and (11) do not contribute to $A(u,\epsilon)$.

loop order	diagram (i)	symmetry factor $S^{(i)}$	group factor $G^{(i)}(n)$
1 loop	(1)	1	n
2 loops	(2)	3	$\frac{1}{3}(n^2+2n)$
3 loops	(3)	36	$\frac{1}{9}(n^3+4n^2+4n)$
	(4)	12	$\frac{1}{3}(n^2+2n)$
4 loops	(5)	432	$\frac{1}{27}(n^4+6n^3+12n^2+8n)$
	(7)	576	$\frac{1}{9}(n^3+4n^2+4n)$
	(8)	288	$\frac{1}{27}(n^3+10n^2+16n)$
5 loops	(9)	5184	$\frac{1}{81}(n^5+8n^4+24n^3+32n^2+16n)$
	(12)	10368	$\frac{1}{27}(n^4+6n^3+12n^2+8n)$
	(13)	6912	$\frac{1}{27}(n^4+6n^3+12n^2+8n)$
	(14)	6912	$\frac{1}{27}(n^4+6n^3+12n^2+8n)$
	(15)	2304	$\frac{1}{9}(n^3+4n^2+4n)$
	(16)	2592	$\frac{1}{81}(n^4+8n^3+32n^2+40n)$
	(17)	20736	$\frac{1}{81}(n^4+12n^3+36n^2+32n)$
	(18)	10368	$\frac{1}{81}(5n^3+32n^2+44n)$

TABLE II. Coefficients $f_i^{(m)}$ of the functions $\tilde{\beta}(u)$, $\zeta_r(u)$ and $\zeta_\phi(u)$ for $i = 1, 2, 3$, respectively, and coefficients $c_{_{Bm}}$ of $B(u)$, compare Eqs. (2.20) and (2.21), for $n = 0, 1, 2, 3$ up to five-loop order ($m = 6$ for $\tilde{\beta}$, $m = 5$ for ζ_r and ζ_ϕ, $m = 4$ for B). For $f_i^{(m)}$ compare Table 1 of Ref. [8].

	$\tilde{\beta}_u$	ζ_r	ζ_φ	B
$n = 0$	0	8	0	0
	32	-80	-16	0
	-672	3552	128	0
	43989.9534	-223152.607	-8000	0
	-4166409.19	18836823.8	500639.112	0
	498653403.0			
$n = 1$	0	12	0	1/2
	36	-120	-24	0
	-816	6048	216	9
	56245.8519	-413813.942	-14040	-761.422836
	-5632017.54	37512804.7	958294.321	44244.7100
	708814936.0			
$n = 2$	0	16	0	1
	40	-160	-32	0
	-960	9024	320	24
	69029.7505	-660870.017	-21120	-2256.06766
	-7268274.40	63662497.1	1566676.68	141294.329
	956636505.0			
$n = 3$	0	20	0	3/2
	44	-200	-40	0

-1104	12480	440	45
82341.6490	-967074.371	-29000	-4653.13955
-9075019.76	98265069.9	2333667.84	310944.846
1243816220.0			

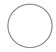
TABLE III. Coefficients $f_u^{(k)}$ of the $\epsilon = 4 - d$ expansion of the fixed point value $u^\star(\epsilon) = \sum_{k=1}^{5} f_u^{(k)} \epsilon^k$ up to $k = 5$ (five-loop order) for $n = 1, 2, 3$.

	k	$f_u^{(k)}$
$n = 1$	1	1/36
	2	17/972
	3	-0.0114632463
	4	0.0223877393
	5	-0.0703070454
$n = 2$	1	1/40
	2	3/200
	3	-0.00896474627
	4	0.0170850033
	5	-0.0492703392
$n = 3$	1	1/44
	2	69/5324
	3	-0.00718789532
	4	0.0134615145
	5	-0.0358025667

TABLE IV. Coefficients $c^{\pm}_{Fm}$ of $F_{+}(u)$ and $F_{-}(u)$ for $n = 1, 2, 3$ defined in Eqs. (3.6) and (3.7), respectively. For c^{+}_{Fm}, m refers to u^m corresponding to $(m+1)$-loop order whereas for c^{-}_{Fm}, m refers to u^{m-1} corresponding to m-loop order.

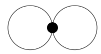
	m	c^{+}_{Fm}	c^{-}_{Fm}
$n = 1$	0	-1	1/2
	1	-6	-4
	2	-22.6976286	72
	3	-722.742494	-5189.75477
	4	34775.5862	433582.588
	5		47754702.4
$n = 2$	0	-2	1/2
	1	-16	-4
	2	-92.5270094	64
	3	-2430.86459	
	4	102469.659	
$n = 3$	0	-3	1/2
	1	-30	-4
	2	-233.488142	56
	3	-5742.02974	
	4	204463.778	

1 loop:



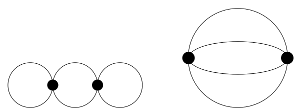
(1)

2 loops:



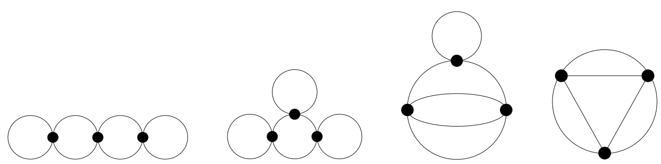
(2)

3 loops:



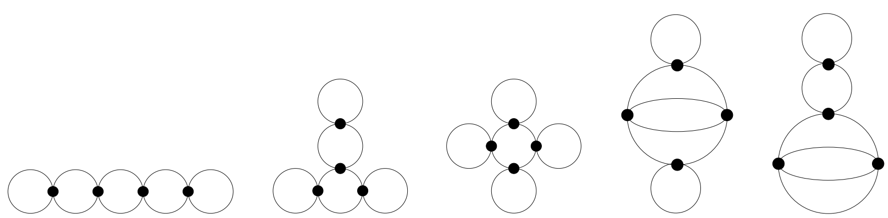
(3) (4)

4 loops:

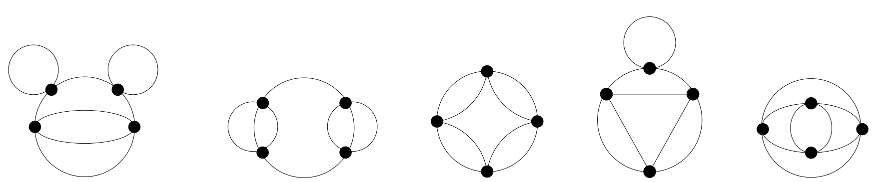


(5) (6) (7) (8)

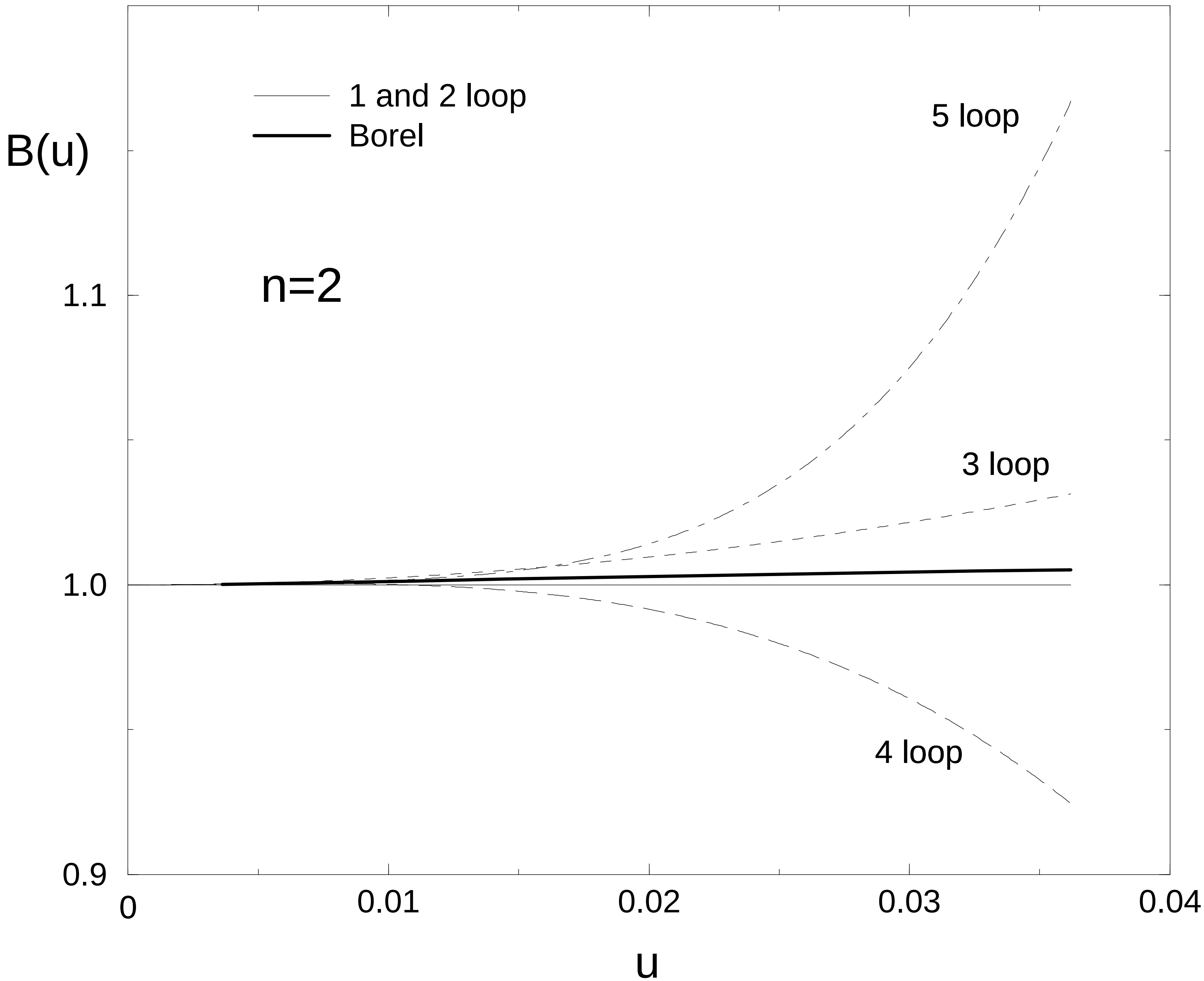
5 loops:



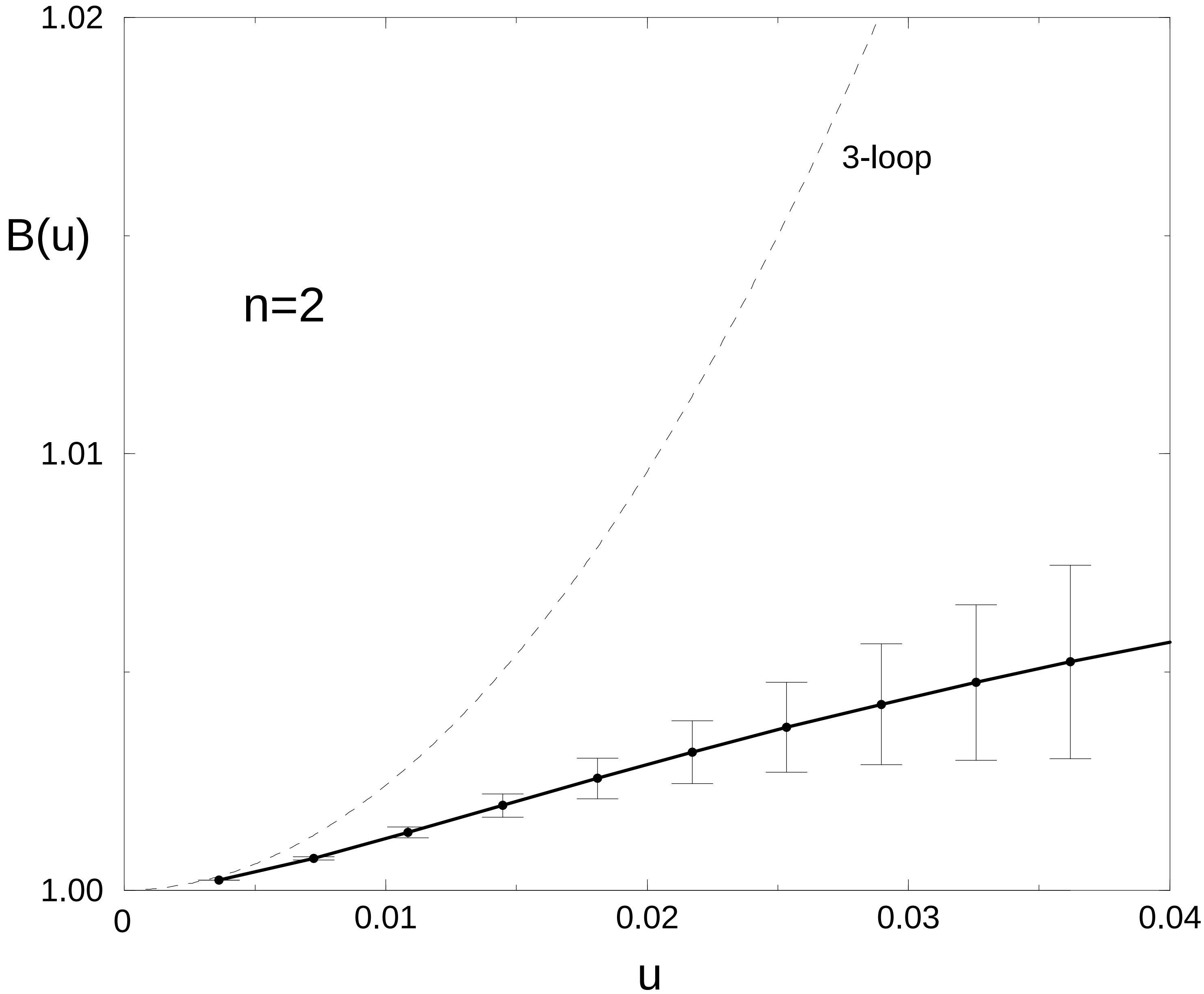
(9) (10) (11) (12) (13)



(14) (15) (16) (17) (18)



1 and 2 loop
Borel
5 loop
B(u)
n=2
1.1
3 loop
1.0
4 loop
0.9
0
0.01
0.02
0.03
0.04
u



1.02
1.01
1.00
B(u)
n=2
3-loop
0
0.01
0.02
0.03
0.04
u